New models for Veneziano amplitudes : Combinatorial, symplectic and supersymmetric aspects

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The bosonic string theory evolved as an attempt to find a physical/quantum mechanical model capable of reproducing Euler's beta function (Veneziano amplitude) and its multidimensional analogue. The multidimensional analogue of beta function was studied mathematically for some time from different angles by mathematicians such as Selberg,Weil and Deligne among many others. The results of their studies apparently were not taken into account in physics literature on string theory. In several recent publications attempts were made to restore the missing links. As discussed in these publications, the existing mathematical interpretation of the multidimensional analogue of Euler's beta function as one of the periods associated with the corresponding differential form "living" on the Fermat-type (hyper) surface, happens to be crucial for restoration of the quantum/statistical mechanical models reproducing such generalized beta function. There is a number of nontraditional models -all interrelated- capable of reproducing the Veneziano amplitudes. In this work we would like to discuss two of such new models: symplectic and supersymmetric. The symplectic model is based on observation that the Veneziano amplitude is just the Laplace transform of the generating function for the Ehrhart polynomial. Such a polynomial counts the number of lattice points inside the rational polytope (i.e. polytope whose vertices are located at the nodes of a regular lattice) and at its boundaries. In the present case the polytope is a regular simplex. It is a deformation retract for the Fermat-type (hyper)surface (perhaps inflated, as explained in the text). Using known connections between polytopes and dynamical systems the quantum mechanical system associated with such a dynamical system is found. The ground state of this system is degenerate with degeneracy factor given by the Ehrhart polynomial. Using some ideas by Atiyah, Bott and Witten we argue that the supersymmetric model related to the symplectic can be recovered. While recovering this model, we demonstrate that the ground state of such a model is degenerate with the same degeneracy factor as for earlier obtained symplectic model. Since the wave functions of this model are in one to one correspondence with the Veneziano amplitudes, this exactly solvable supersymmetric (and, hence, also symplectic) model is sufficient for recovery of the partition function reproducing the Veneziano amplitudes thus providing the exact solution of the Veneziano model.

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1 Introduction

In 1968 Veneziano [1] postulated the 4-particle scattering amplitude $A(s,t,u)$ given (up to a common constant factor) by

$$A(s,t,u) = V(s,t) + V(s,u) + V(t,u), \qquad (1)$$

where

$$V(s,t) = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx \equiv B(-\alpha(s),-\alpha(t)) \qquad (2)$$

is the Euler beta function and $\alpha(x)$ is the Regge trajectory usually written as $\alpha(x) = \alpha(0)+\alpha'x$ with $\alpha(0)$ and α' being the Regge slope and intercept, respectively. In case of space-time metric with signature $\{-,+,+,+\}$ the Mandelstam variables s, t and u entering the Regge trajectory are defined by [2]

$$s = -(p_1+p_2)^2; \; t = -(p_2+p_3)^2; \; u = -(p_3+p_1)^2. \qquad (3)$$

The 4-momenta p_i are constrained by the energy-momentum conservation law leading to relation between the Mandelstam variables:

$$s + t + u = \sum_{i=1}^4 m_i^2. \qquad (4)$$

Veneziano [1] noticed[2] that to fit experimental data the Regge trajectories should obey the constraint

$$\alpha(s) + \alpha(t) + \alpha(u) = -1 \qquad (5)$$

consistent with Eq.(4) in view of the definition of $\alpha(s)$. The Veneziano condition, Eq.(5), can be rewritten in a more general form. Indeed, let m,n,l be some integers such that $\alpha(s)m + \alpha(t)n + \alpha(u)l = 0$. Then by adding this equation to Eq.(5) we obtain, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} = -1$, or more generally, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} + \tilde{k}\cdot 1 = 0$. Both equations have been studied extensively in the book by Stanley [3] from the point of view of commutative algebra, polytopes, toric varieties, invariants of finite groups, etc. This observation plays the major role in developments we would like to present in this work and elsewhere.

In 1967-a year before Veneziano's paper was published- the paper [4] by Chowla and Selberg appeared relating Euler's beta function to the periods of elliptic integrals. The result by Chowla and Selberg was generalized by Andre

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[2]To get our Eq.(5) from Eq.7 of Veneziano paper, it is sufficient to notice that his $1 - \alpha(s)$ corresponds to ours $-\alpha(s)$.

Weil whose two influential papers [5,6] brought into the picture the periods of Jacobians of the Abelian varieties, Hodge rings, etc. Being motivated by these papers, Benedict Gross wrote a paper [7] in which the beta function appears as period associated with the differential form "living" on the Jacobian of the Fermat curve. His results as well as those by Rohrlich (placed in the appendix to Gross paper) have been subsequently documented in the book by Lang [8]. Although in the paper by Gross the multidimensional extension of beta function is considered only briefly, e.g.[7,p.207], the computational details were not provided hovever. These computational details can be found in our recently published papers, Refs.[9,10]. To obtain the multidimensional extension of beta function as period integral, following logic of papers by Gross and Deligne [11], one needs to replace the Fermat curve by the Fermat hypersufrace, to embed it into the complex projective space, and to treat it as Kähler manifold. The differential forms living on such manifold are associated with periods of the Fermat hypersurface. Physical considerations require this Kähler manifold to be of the Hodge type. In his lecture notes [11] Deligne noticed that the Hodge theory needs some essential changes (e.g. mixed Hodge structures, etc.) if the Hodge-Kähler manifolds possess singularities. Such modifications may be needed upon development of our formalism. A monograph by Carlson et al, Ref.[12], contains an up to date exaustive information regarding such modifications, etc. Fortunately, to obtain the multiparticle Veneziano amplitudes these complications are not nesessary. In Ref. [10] we demonstrated that the period integrals living on the Fermat hypersurfaces, when properly interpreted, provide the tachyon-free (Veneziano-like) multiparticle amplitudes whose paricle spectrum reproduces those known for both the open and closed bosonic strings. Naturally, the question arises: If this is so, then what kind of a model is capable of reproducing such amplitudes ? In this paper we would like to discuss some combinatorial properties of the Veneziano amplitudes (easily extendable to the Veneziano-like) sufficient for reproducing at least two of such models: symplectic and supersymmetric. Mathematically, the results presented below are in accord with those by Vergne [13] whose work does not contain practical applications.

This work is organized as follows. In Section 2 we explain the combinatorial nature of the Veneziano amplitudes by connecting them with the generating function for the Ehrhart polynomial. Such a polynomial counts the number of points inside the rational polytope (i.e. polytope whose vertices are located at the nodes of the regular $k-$dimensional lattice) and at its boundaries (faces). In the present case the polytope is a regular simplex which is a deformation retract for the Fermat-type (hyper) surface living in the complex projective space [9,10]. The connections between the polytopes and dynamical systems are well-known [13,14]. Development of these connections is presented in Sections 2-4 where we find the corresponding quantum mechanical system whose ground state is degenerate with degeneracy factor being identified with the Ehrhart polynomial. The obtained result is in accord with that by Vergne [13]. In addition, in Section 5 the generating function for the Ehrhart polynomial is reinterpreted in terms of the Poincare' polynomial. Such a polynomial is used, for instance, in the

theory of invariants of finite (pseudo)reflection groups [3,15]. Obtained indentification reveals the topological and group-theoretic nature of the Veneziano amplitudes. To strengthen this point of view, we use some results by Atiyah and Bott [16] inspired by still earlier work by Witten [17] on supersymmetric quantum mechanics. They allow us to think about the Veneziano amplitudes using therminology of intersection theory [18]. This is consistent with earlier mentioned interpretation of these amplitudes in terms of periods of the Fermat (hyper)surface [19]. It also makes computation of these amplitudes analogous to those for the Witten -Kontsevich model [20, 21], whose refinements can be found in our earlier work, Ref.[22]. For the sake of space, in this work we do not develop these connections with the Witten- Kontsevich model any further. Instead, we discuss the supersymmetric model associated with symplectic model described earlier and treat it with help of the Lefshetz isomorphism theorem. This allows us to look at the problem of computation of the spectrum for such a model from the point of view of the theory of representations of the complex semisimple Lie algebras. Using some results by Serre [23] and Ginzburg [24] we demonstrate that the ground state for such finite dimensional supersymmetric quantum mechanical model is degenerate with degeneracy factor coinciding with the Erhardt polynomial. This result is consistent with that obtained in Section 4 by different methods.

2 The Veneziano amplitude and the Ehrhart polynomial

In view of Eq.(2), consider an identity taken from [25],

$$\frac{1}{(1-tz_0)\cdots(1-tz_k)} = (1 + tz_0 + (tz_0)^2 + ...) \cdots (1 + tz_n + (tz_n)^2 + ...)$$

$$= \sum_{n=0}^{\infty} (\sum_{k_0+...+k_k=n} z_0^{k_0} \cdots z_k^{k_k}) t^n. \tag{6}$$

When $z_0 = ... = z_k = 1$,the inner sum in the last expression provides the total number of monomials of the type $z_0^{k_0} \cdots z_k^{k_k}$ with $k_0 + ... + k_k = n$. The total number of such monomials is given by the binomial coefficient[3]

$$p(k,n) = \frac{(k+n)!}{k!n!} = \frac{(n+1)(n+2)\cdots(n+k)}{k!} = \frac{(k+1)(k+2)\cdots(k+n)}{n!}. \tag{7}$$

For this special case (6) is converted to a useful expansion,

$$P(k,t) \equiv \frac{1}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} p(k,n)t^n. \tag{8}$$

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[3]The reason for displaying 3 different forms of the same combinatorial factor will be explained shortly below.

In view of the integral representation of the beta function given by (2), we replace $k + 1$ by $\alpha(s) + 1$ in (8) and use it in the beta function representation of $V(s, t)$. Straightforward calculation produces the following known result [2]:

$$V(s, t) = -\sum_{n=0}^{\infty} p(\alpha(s), n) \frac{1}{\alpha(t) - n}. \tag{9}$$

The r.h.s. of (9) is effectively the Laplace transform of the generating function (8) which can be interpreted as a partition function in the sence of statistical mechanics. The purpose of this Letter is to demonstrate that such an interpretation is not merely a conjecture and, in view of this, to find the statisical mechanical/quantum model whose partition function is given by Eq.(8). Our arguments are not restricted to the 4-particle amplitude. Indeed, as we argued earlier [10], the multidimensional extension of Euler's beta function producing murtiparticle Veneziano amplitudes (upon symmetrization analogous to the 4-particle case) is given by the following integral attributed to Dirichlet

$$\mathcal{D}(x_1, ..., x_k) = \int\limits_{\substack{u_1 \geq 0, ..., u_k \geq 0 \\ u_1 + \cdots + u_k \leq 1}} \int u_1^{x_1 - 1} u_2^{x_2 - 1} ... u_k^{x_k - 1} (1 - u_1 - ... - u_k)^{x_{k+1} - 1} du_1 ... du_k. \tag{10}$$

In this integral let $t = u_1 + ... + u_k$. This allows us to use already familiar expansion (8). In addition, the following identity

$$t^n = (u_1 + ... + u_k)^n = \sum_{n=(n_1, ..., n_k)} \frac{n!}{n_1! n_2! ... n_k!} u_1^{n_1} \cdots u_k^{n_k} \tag{11}$$

with restriction $n = n_1 + ... + n_k$ is of importance as well. This type of identity was used earlier in our work on Kontsevich-Witten model [22]. Moreover, from the same paper it follows that the above result can be presented as well in the alternative useful form:

$$(u_1 + ... + u_k)^n = \sum_{\lambda \vdash k} f^\lambda S_\lambda(u_1, ..., u_k), \tag{12}$$

where the Schur polynomial S_λ is defined by

$$S_\lambda(u_1, ..., u_k) = \sum_{n=(n_1, ..., n_k)} K_{\lambda, n} u_1^{n_1} \cdots u_k^{n_k} \tag{13}$$

with coefficients $K_{\lambda, n}$ known as Kostka numbers, f^λ being the number of standard Young tableaux of shape λ and the notation $\lambda \vdash k$ meaning that λ is partition of k. Through such a connection with Schur polynomials one can develop connections with the Kadomtsev-Petviashvili (KP) hierarchy of nonlinear exactly integrable systems on one hand and with the theory of Schubert varieties on another. Although details can be found in our earlier work [22], in this Letter we shall discuss these issues a bit further in Section 5. Use of (11)

6

in (10) produces, after performing the multiple Laplace transform, the following part of the multiparticle Veneziano amplitude

$$A(1, ...k) = \frac{\Gamma_{n_1...n_k}(\alpha(s_{k+1}))}{(\alpha(s_1) - n_1) \cdots (\alpha(s_k) - n_k)}. \tag{14}$$

Even though the residue $\Gamma_{n_1...n_k}(\alpha(s_{k+1}))$ contains all the combinatorial factors, the obtained result should still be symmetrized (in accord with the 4-particle case considered by Veneziano) in order to obtain the full murtiparticle Veneziano amplitude. Since in the above general multiparticle case the same expansion (8) was used, for the sake of space it is sufficient to focus on the 4-particle amplitude only. This task is reduced to further study of the expansion (8). Such an expansion can be looked upon from several different angles. For instance, we have mentioned already that it can be interpreted as a partition function. In addition, it is the generating function for the Ehrhart polynomial. The combinatorial factor $p(k, n)$ defined in (7) is the simplest example of the Ehrhart polynomial. Evidently, it can be written formally as

$$p(k, n) = a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0. \tag{15}$$

Let $\mathcal{P}$ be *any* convex rational polytope that is the polytope whose vertices are located at the nodes of some $n-$dimensional $\mathbf{Z}^n$ lattice. Then, the Ehrhart polynomial for the inflated polytope $\mathcal{P}$ (with coefficient of inflation $k = 1, 2, ...$) can be written as

$$|k\mathcal{P} \cap \mathbf{Z}^n| = \mathfrak{P}(k, n) = a_n(\mathcal{P})k^n + a_{n-1}(\mathcal{P})k^{n-1} + \cdots + a_0(\mathcal{P}) \tag{16}$$

with coefficients $a_0, ..., a_n$ being specific for a given type of polytope $\mathcal{P}$. In the case of Veneziano amplitude the polynomial $p(k, n)$ counts number of points inside the $n-$dimensional inflated simplex (with inflation coefficient $k = 1, 2, ...$). Irrespective to the polytope type, it is known [26] that $a_0 = 1$ and $a_n = Vol\mathcal{P}$, where $Vol\mathcal{P}$ is the *Euclidean* volume of the polytope. These facts can be easily checked for $p(k, n)$. To calculate the remaining coefficients of such polynomial explicitly for arbitrary convex rational polytope $\mathcal{P}$ is a difficult task in general. Such a task was accomplished only recently in [27]. The authors of [27] recognized that in order to obtain the remaining coefficients, it is useful to calculate the generating function for the Ehrhart polynomial. Long before the results of [27] were published, it was known [3,15], that the generating function for the Ehrhart polynomial of $\mathcal{P}$ can be written in the following universal form

$$\mathcal{F}(\mathcal{P}, x) = \sum_{k=0}^{\infty} \mathfrak{P}(k, n)x^k = \frac{h_0(P) + h_1(P)x + \cdots + h_n(P)x^n}{(1 - x)^{n+1}} \tag{17}$$

The above general result might be of some use in case of possible generalizations of the Veneziano amplitudes and the associated with them partition functions. Its mathematical meaning is discussed further in Section 5.

The fact that the combinatorial factor $p(k, n)$ in (7) can be formally written in several equivalent ways has some physical significance. For instance, in

7

particle physics literature, e.g. see [2], the second option is commonly used. Let us recall how this happens. One is looking for an expansion of the factor $(1-x)^{-\alpha(t)-1}$ under the integral of beta function, e.g. see Eq.(2). Looking at Eq.(17) one realizes that the Mandelstam variable $\alpha(t)$ plays the role of dimensionality of **Z-** lattice. Hence, in view of (6), we have to identify it with n in the second option provided by (7). This is not the way such an identification is done in physics literature where, in fact, the third option in (7) is commonly used with $n = \alpha(t)$ being effectively the inflation factor while k is effectively the dimensionality of the lattice.[4] A quick look at Eq.s(8) and (17) shows that under such circumstances the generating function for the Ehrhart polynomial and that for the Veneziano amplitude are formally not the same. In the first case one is dealing with lattices of *fixed* dimensionality and considering summation over various inflation factors at the same time. In the second (physical) case, one is dealing with the *fixed inflation factor* $n = \alpha(t)$ while summing over lattices of different dimensionalities. Such arguments are superficial however in view of Eq.s(6) and (17) above.Using these equations it is clear that mathematically correct agreement between Eq.s(8) and (17) can be reached if one is using $\mathfrak{P}(k,n) = p(k,n)$ with the second option taken from (7). By doing so no changes in the pole locations for the Veneziano amplitude occur. Moreover, for a given pole the second and the third option in (7) produce exactly the same contributions into the residue thus making them physically indistinguishable. The interpretation of the Veneziano amplitude as the Laplace transform of the Ehrhart polynomial generating function provides a very compelling reason for development of the alternative string-theoretic formalism. In addition, it allows us to think about possible generalizations of the Veneziano amplitude using generating functions for the Ehrhart polynomials for polytopes other than the $n-$dimensional inflated simplex used for the Veneziano amplitudes. As it is demonstrated by Stanley [3,15], Eq.(17) has a group invariant meaning as the Poincare$'$ polynomial for the so called Stanley-Reisner polynomial ring.[5] From the same reference one can also find connections of these results with toric varieties. In view of Ref. [14], this observation is sufficient for restoration of physical models reproducing the Veneziano and Veneziano-like amplitudes. Thus, in the rest of this paper we shall discuss some approaches to the design of these models. Space limitations forbid us from discussing other directions. They will be discussed in the subsequent publications.

3   Motivating examples

To facilitate our readers understanding, we would like to illustrate general principles using simple examples. We begin by considering a finite geometric pro-

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[4]We have to warn our readers that nowhere in physics literature such combinatorial terminology is used.

[5]In Section 5 we provide some additional details on this topic.

gression of the type

$$\mathcal{F}(c,m) = \sum_{l=-m}^{m} \exp\{cl\} = \exp\{-cm\} \sum_{l=0}^{\infty} \exp\{cl\} + \exp\{cm\} \sum_{l=-\infty}^{0} \exp\{cl\}$$

$$= \exp\{-cm\} \frac{1}{1-\exp\{c\}} + \exp\{cm\} \frac{1}{1-\exp\{-c\}}$$

$$= \exp\{-cm\} \left[\frac{\exp\{c(2m+1)\}-1}{\exp\{c\}-1} \right]. \tag{18}$$

The reason for displaying the intermediate steps will be explained shortly below. First, however, we would like to consider the limit : $c \to 0^+$ of $\mathcal{F}(c,m)$. It is given by $\mathcal{F}(0,m) = 2m+1$. The number $2m+1$ equals to the number of integer points in the segment $[-m,m]$ *including boundary* points. It is convenient to rewrite the above result in terms of $x = \exp\{c\}$ so that we shall write formally $\mathcal{F}(x,m)$ instead of $\mathcal{F}(c,m)$ from now on. Using such notation, consider a related function

$$\bar{\mathcal{F}}(x,m) = (-1)\mathcal{F}(\frac{1}{x}, -m). \tag{19}$$

This type of relation (the *Ehrhart-Macdonald reciprocity law*) is characteristic for the Ehrhart polynomial for rational polytopes discussed earlier. In the present case we obtain,

$$\bar{\mathcal{F}}(x,m) = (-1)\frac{x^{-(-2m+1)}-1}{x^{-1}-1}x^m. \tag{20}$$

In the limit $x \to 1+0^+$ we obtain : $\bar{\mathcal{F}}(1,m) = 2m-1$. The number $2m-1$ is equal to the number of integer points strictly *inside* the segment $[-m,m]$. Both $\mathcal{F}(0,m)$ and $\bar{\mathcal{F}}(1,m)$ provide the simplest possible examples of the Ehrhart polynomials if we identify m with the inflation factor k.

These, seemingly trivial, results can be broadly generalized. First, we replace x by $\mathbf{x} = x_1 \cdots x_d$, next we replace the summation sign in the left hand side of Eq.(18) by the multiple summation, etc. Thus obtained function $\mathcal{F}(\mathbf{x},m)$ in the limit $x_i \to 1+0^+$, $i = 1-d$, produces the anticipated result : $\mathcal{F}(\mathbf{1},m) = (2m+1)^d$. It describes the number of points inside and at the faces of a $d-$ dimensional cube in the Euclidean space $\mathbf{R}^d$. Accordingly, for the number of points strictly inside the cube we obtain : $\bar{\mathcal{F}}(\mathbf{1},m) = (2m-1)^d$.

To move further, we would like to remind to our readers that mathematically a subset of $\mathbf{R}^n$ is considered to be a *polytope (or polyhedron)* $\mathcal{P}$ if there is a $r \times d$ matrix $\mathbf{M}$ (with $r \le d$) and a vector $\mathbf{b} \in \mathbf{R}^d$ such that $\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^d \mid \mathbf{Mx} \le \mathbf{b}\}$. Provided that the Euclidean d-dimensional scalar product is given by $< \mathbf{x} \cdot \mathbf{y} > = \sum_{i=1}^{d} x_i y_i$, a *rational (or integral) polytope (or polyhedron)* $\mathcal{P}$ is defined by the set

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^d \mid < \mathbf{a}_i \cdot \mathbf{x} > \le \beta_i , i = 1, ..., r\} \tag{21}$$

9

where $\mathbf{a}_i \in \mathbf{Z}^n$ and $\beta_i \in \mathbf{Z}$ for $i = 1, ..., r$. Let $Vert\mathcal{P}$ denote the vertex set of the rational polytope, in the case considered thus far, the $d-$dimensional cube. Let $\{u_1^v, ..., u_d^v\}$ denote the orthogonal basis (not necessarily of unit length) made of the highest weight vectors of the Weyl-Coxeter reflection group B_d appropriate for the cubic symmetry [28].These vectors are oriented along the positive semi axes with respect to the center of symmetry of the cube. When parallel translated to the edges ending at particular hypercube vertex $\mathbf{v}$, they can point either in or out of this vertex. Then, the d-dimensional version of Eq.(18) can be rewritten in notations just introduced as follows

$$\sum_{\mathbf{x} \in \mathcal{P} \cap \mathbf{Z}^d} \exp\{< \mathbf{c} \cdot \mathbf{x} >\} = \sum_{\mathbf{v} \in Vert\mathcal{P}} \exp\{< \mathbf{c} \cdot \mathbf{v} >\} \left[\prod_{i=1}^{d} (1 - \exp\{-c_i u_i^v\}) \right]^{-1}.$$
(22)

The correctness of this equation can be readily checked by considering special cases of a segment, square, cube, etc. The result, Eq.(22), obtained for the polytope of cubic symmetry can be extended to the arbitrary convex centrally symmetric polytope. Details can be found in Ref.[29]. Moreover, the requirement of central symmetry can be relaxed to the requirement of the convexity of $\mathcal{P}$ only. In such general form the relation given by Eq.(22) was obtained by Brion [30]. It is of central importance for the purposes of this work: the limiting procedure $c \to 0^+$ produces the number of points inside (and at the boundaries) of the polyhedron $\mathcal{P}$ in the l.h.s. of Eq.(22) and, if the polyhedron is rational and inflated, this procedure produces the Ehrhart polynomial. Actual computations are done with help of the r.h.s. of Eq.(22) as will be demonstrated below.

4 The Duistermaat-Heckman formula and the

Khovanskii-Pukhlikov correspondence

Since the description of the Duistermaat-Heckman (D-H) formula can be found in many places, we would like to be brief in discussing it now in connection with earlier obtained results. Let $M \equiv M^{2n}$ be a compact symplectic manifold equipped with the moment map $\Phi : M \to \mathbf{R}$ and the (Liouville) volume form $dV = \left(\frac{1}{2\pi}\right)^n \frac{1}{n!} \Omega^n$. According to the Darboux theorem, locally $\Omega = \sum_{l=1}^{n} dq_l \wedge dp_l$. We expect that such a manifold has isolated fixed points p belonging to the fixed point set $\mathcal{V}$ associated with the isotropy subgroup of the group G acting on M. Then, in its most general form, the D-H formula can be written as [13,14,31]

$$\int_M dV e^{\Phi} = \sum_{p \in \mathcal{V}} \frac{e^{\Phi(p)}}{\prod_j a_{j,p}}$$
(23)

where $a_{1,p}, ..., a_{n,p}$ are the weights of the linearized action of G on $T_p M$. Using Morse theory, Atiyah [32] and others, e.g. see Ref.[14] for additional references, have demonstrated that it is sufficient to keep terms up to quadratic in the

10

expansion of Φ around given p. In such a case the moment map can be associated with the Hamiltonian for the finite set of harmonic oscillators. In the properly chosen system of units the coefficients $a_{1,p}, ..., a_{n,p}$ are just "masses" m_i of the individual oscillators. Unlike truly physical masses, some of $m'_i s$ can be negative.

Based on the information just provided, we would like be more specific now. To this purpose, following Vergne [33] and Brion [30], we would like to consider the D-H integral of the form

$$I(k \; ; \; y_1, y_2) = \int_{k\Delta} dx_1 dx_2 \exp\{-(y_1 x_1 + y_2 x_2)\}, \tag{24}$$

where $k\Delta$ is the standard dilated simplex with dilation coefficient k^6. Calculation of this integral can be done exactly with the result:

$$I(k \; ; \; y_1, y_2) = \frac{1}{y_1 y_2} + \frac{e^{-ky_1}}{y_1(y_1 - y_2)} + \frac{e^{-ky_2}}{y_2(y_2 - y_1)} \tag{25}$$

consistent with Eq.(23). In the limit: $y_1, y_2 \to 0$ some calculation produces the anticipated result : $Vol k\Delta = k^2/2!$ for the *Euclidean* volume of the dilated simplex. Next, to make a connection with the previous section, in particular, with Eq.(22), consider the following sum

$$S(k \; ; y_1, y_2) = \sum_{(l_1, l_2) \in k\Delta} \exp\{-(y_1 l_1 + y_2 l_2)\}$$
$$= \frac{1}{1 - e^{-y_1}} \frac{1}{1 - e^{-y_2}} + \frac{1}{1 - e^{y_1}} \frac{e^{-ky_1}}{1 - e^{y_1 - y_2}} + \frac{1}{1 - e^{y_2}} \frac{e^{-ky_2}}{1 - e^{y_2 - y_1}} \tag{26}$$

related to the D-H integral, Eq.(24). Its calculation will be explained momentarily. In spite of the connection with the D-H integral, the limiting procedure: $y_1, y_2 \to 0$ in the last case is much harder to perform. It is facilitated by use of the following expansion

$$\frac{1}{1 - e^{-s}} = \frac{1}{s} + \frac{1}{2} + \frac{s}{12} + O(s^2). \tag{27}$$

Rather lenghty calculations produce the anticipated result : $S(k \; ; 0,0) = k^2/2! + 3k/2 + 1 \equiv |k\Delta \cap Z^2| \equiv \mathfrak{P}(k, 2)$ for the Ehrhart polynomial. Since generalization of the obtained results to simplicies of higher dimensions is straightforward, the relevance of these results to the Veneziano amplitude should be evident. To make it more explicit we have to make several steps still. First, we would like to explain how the result (26) was obtained. By doing so we shall gain some

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[6]Our choice of the simplex as the domain of integration is caused by our earlier made observation [10] that the deformation retract of the Fermat (hyper)surface (on which the Veneziano amplitude lives) is just the standard simplex. Since such Fermat surface is a complex Kähler-Hodge type manifold and since all Kähler manifolds are symplectic [14,34], our choice makes sense.

additional physical information. Second, we would like to explain in some detail the connection between the integral (25), and the sum (26). Such a connection is made with help of the Khovanskii-Pukhlikov correspondence.

We begin with calculations of the sum, Eq.(26). To do this we need a definition of the convex rational polyhedral cone σ. It is given by

$$\sigma = \mathbf{Z}_{\geq 0} a_1 + \cdots + \mathbf{Z}_{\geq 0} a_d \ , \tag{28}$$

where the set $a_1, ..., a_d$ forms a basis (not nesessarily orthogonal) of the d-dimensional vector space V, while $\mathbf{Z}_{\geq 0}$ are non negative integers. It is known that all combinatorial information about the polytope $\mathcal{P}$ is encoded in the *complete* fan made of cones whose apexes all having the same origin in common. Details can be found in literature [14,18]. At the same time, the vertices of $\mathcal{P}$ are also the apexes of the respective cones. Following Brion[30] this fact allows us to write the l.h.s. of Eq.(22) as

$$f(\mathcal{P}, x) = \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbf{Z}^d} \mathbf{x}^{\mathbf{m}} = \sum_{\sigma \in Vert\mathcal{P}} \mathbf{x}^{\sigma} \tag{29}$$

so that for the *dilated* polytope the above statement reads as follows [30,35]:

$$f(k\mathcal{P}, x) = \sum_{\mathbf{m} \in k\mathcal{P} \cap \mathbf{Z}^d} \mathbf{x}^{\mathbf{m}} = \sum_{i=1}^{n} \mathbf{x}^{k\mathbf{v}_i} \sum_{\sigma_i} \mathbf{x}^{\sigma_i}. \tag{30}$$

In the last formula the summation is taking place over all vertices whose location is given by the vectors from the set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. This means that in actual calculations one can first calculate the contributions coming from the cones σ_i of the undilated (original) polytope $\mathcal{P}$ and only then one can use the last equation in order to get the result for the dilated polytope.

Let us apply these general results to our specific problem of computation of $S(k \ ; \ y_1, y_2)$ in Eq.(26). We have our simplex with vertices in x-y plane given by the vector set $\{ \mathbf{v}_1 = (0,0), \mathbf{v}_2 = (1,0), \mathbf{v}_3 = (0,1)\}$, where we have written the x coordinate first. In this case we have 3 cones: $\sigma_1 = l_2 \mathbf{v}_2 + l_3 \mathbf{v}_3$, $\sigma_2 = \mathbf{v}_2 + l_1(-\mathbf{v}_2) + l_2(\mathbf{v}_3 - \mathbf{v}_2); \sigma_3 = \mathbf{v}_3 + l_3(\mathbf{v}_2 - \mathbf{v}_3) + l_1(-\mathbf{v}_3); \{l_1, l_2 , l_3 \} \in \mathbf{Z}_{\geq 0}$. In writing these expressions for the cones we have taken into account that, according to Brion, when making calculations the apex of each cone should be chosen as the origin of the coordinate system. Calculation of contributions to the generating function coming from σ_1 is the most straightforward. Indeed, in this case we have $\mathbf{x} = x_1 x_2 = e^{-y_1} e^{-y_2}$. Now, the symbol $\mathbf{x}^{\sigma}$ in Eq.s(29) should be understood as follows. Since σ_i , $i = 1 - 3$, is actually a vector, it has components, like those for $\mathbf{v}_1$, etc. We shall write therefore $\mathbf{x}^{\sigma} = x_1^{\sigma(1)} \cdots x_d^{\sigma(d)}$ where $\sigma(i)$ is the i-th component of such a vector. Under these conditions calculation of the contributions from the first cone with the apex located at (0,0) is completely straightforward and is given by

$$\sum_{(l_2, l_3) \in Z_+^2} x_1^{l_2} x_2^{l_3} = \frac{1}{1 - e^{-y_1}} \frac{1}{1 - e^{-y_2}}. \tag{31}$$

12

It is reduced to the computation of the infinite geometric progression. But physically, the above result can be looked upon as a product of two partition functions for two harmonic oscillators whose ground state energy was discarded. By doing the rest of calculations in the way just described we reobtain $S(k ; y_1, y_2)$ from Eq.(26) as required. This time, however, we know that the obtained result is associated with the assembly of harmonic oscillators of frequencies $\pm y_1$ and $\pm y_2$ and $\pm(y_1 - y_2)$ whose ground state energy is properly adjusted. The "frequencies" (or masses) of these oscillators are coming from the Morse-theoretic considerations for the moment maps associated with the critical points of symplectic manifolds as explained in the paper by Atiyah [32]. These masses enter into the "classical" D-H formula, Eq.s(24),(25). It is just a classical partition function for a system of such described harmonic oscillators living in the phase space containing critical points. The D-H classical partition function, Eq.(25), has its quantum analog, Eq.(26), just described. The ground state for such a quantum system is degenerate with the degeneracy being described by the Ehrhart polynomial $\mathfrak{P}(k, 2)$. Such a conclusion is in formal accord with results of Vergne [13].

Since (by definition) the coefficient of dilation *k=1,2,...* , *there is no dynamical system (and its quanum analog) for k=0. But this condition is the condition for existence of the tachyon pole in the Veneziano amplitude, Eq.(2). Hence, in view of the results just described this pole should be considered as unphysical and discarded.* Such arguments are independent of the analysis made in Ref.[10] where the unphysical tachyons are removed with help of the properly adjusted phase factors. Clearly, such factors can be reinstated in the present case as well since their existence is caused by the requirements of the torus action invariance of the Veneziano-like amplitudes as explained in [10,14]. Hence, their presence is consistent with results just presented.

Now we are ready to discuss the Khovanskii-Pukhlikov correspondence. It can be understood based on the following generic example taken from Ref.[36]. We would like to compare the integral

$$I(z) = \int\limits_s^t dx e^{zx} = \frac{e^{tz}}{z} - \frac{e^{sz}}{z} \ \text{ with the sum } S(z) = \sum_{k=s}^t e^{kz} = \frac{e^{tz}}{1 - e^{-z}} + \frac{e^{sz}}{1 - e^z}$$

To do so, following Refs[36-38] we introduce the Todd operator (transform) via

$$Td(z) = \frac{z}{1 - e^{-z}}. \tag{32}$$

Then, it can be demonstrated that

$$Td(\frac{\partial}{\partial h_1}) Td(\frac{\partial}{\partial h_2})(\int_{s-h_1}^{t+h_2} e^{zx} dx) \mid_{h_1=h_1=0} = \sum_{k=s}^t e^{kz}. \tag{33}$$

This result can be now broadly generalized. Following Khovanskii and Pukhlikov

[35], we notice that

$$Td(\frac{\partial}{\partial \mathbf{z}}) \exp\left(\sum_{i=1}^{n} p_i z_i\right) = Td(p_1, ..., p_n) \exp\left(\sum_{i=1}^{n} p_i z_i\right) \qquad (34)$$

By applying this transform to

$$i(x_1, ..., x_k; \xi_1, ..., \xi_k) = \frac{1}{\xi_1 \cdots \xi_k} \exp(\sum_{i=1}^{k} x_i \xi_i) \qquad (35)$$

we obtain,

$$s(x_1, ..., x_k; \xi_1, ..., \xi_k) = \frac{1}{\prod_{i=1}^{k} (1 - \exp(-\xi_i))} \exp(\sum_{i=1}^{k} x_i \xi_i). \qquad (36)$$

This result should be compared now with the individual terms on the r.h.s. of Eq.(22) on one hand and with the individual terms on the r.h.s of Eq.(23) on another. Evidently, with help of the Todd transform the exact "classical" results for the D-H integral are transformed into the "quantum" results of the Brion's identity (22) which is actually equivalent to the Weyl character formula [28].

We would like to illustrate these general observations by comparing the D-H result, Eq.(25), with the Weyl character formula result, Eq.(26). To this purpose we need to use already known data for the cones σ_i, $i = 1 - 3$, and the convention for the symbol $\mathbf{x}^\sigma$. In particular, for the first cone we have already : $\mathbf{x}^{\sigma_1} = x_1^{l_1} x_2^{l_2} = [\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)]$ [7]. Now we assemble the contribution from the first vertex using Eq.(25). We obtain, $[\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)] / y_1 y_2$. Using the Todd transform we obtain as well,

$$Td(\frac{\partial}{\partial l_1}) Td(\frac{\partial}{\partial l_2}) \frac{1}{y_1 y_2} [\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)] \mid_{l_1 = l_2 = 0} = \frac{1}{1 - e^{-y_1}} \frac{1}{1 - e^{-y_2}}. \qquad (37)$$

Analogously, for the second cone we obtain: $\mathbf{x}_2^\sigma = e^- k y_1 e^- l_1 y_1 e^- l_2 (y_1 - y_2)$ so that use of the Todd transform produces:

$$Td(\frac{\partial}{\partial l_1}) Td(\frac{\partial}{\partial l_2}) \frac{1}{y_1 (y_1 - y_2)} e^{-ky_1} e^{-l_1 y_1} e^{-l_2(y_1 - y_2)} \mid_{l_1 = l_2 = 0} = \frac{1}{1 - e^{y_1}} \frac{e^{-ky_1}}{1 - e^{y_1 - y_2}}, \qquad (38)$$

etc.

The obtained results can now be broadly generalized. To this purpose we can formally rewrite the partition function, Eq.(24), in the following symbolic form

$$I(k, \mathbf{f}) = \int_{k\Delta} d\mathbf{x} \exp(-\mathbf{f} \cdot \mathbf{x}) \qquad (39)$$

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[7]To obtain correct results we needed to change signs in front of l_1 and l_2. The same should be done for other cones as well.

valid for any finite dimension d. Since we have performed all calculations explicitly for two dimensional case, for the sake of space, we only provide the idea behind such type of calculation[8]. In particular, using (25) we can rewrite this integral formally as follows

$$\int\limits_{k\Delta} d\mathbf{x}\exp\left(-\mathbf{f}\cdot\mathbf{x}\right) = \sum_p \frac{\exp(-\mathbf{f}\cdot\mathbf{x}(p))}{\prod\limits_i^d h_i^p(\mathbf{f})} \tag{40}$$

Applying the Todd operator (transform) to both sides of this formal expression and taking into account Eq.s(37),(38) (providing assurance that such an operation indeed is legitimate and makes sense) we obtain,

$$\int\limits_{k\Delta} d\mathbf{x}\prod_{i=1}^d \frac{x_i}{1-\exp(-x_i)}\exp\left(-\mathbf{f}\cdot\mathbf{x}\right) = \sum_{\mathbf{v}\in Vert\mathcal{P}} \exp\{<\mathbf{f}\cdot\mathbf{v}>\}\left[\prod_{i=1}^d(1-\exp\{-h_i^v(\mathbf{f})u_i^v\})\right]^{-1}$$
$$= \sum_{\mathbf{x}\in\mathcal{P}\cap\mathbf{Z}^d} \exp\{<\mathbf{f}\cdot\mathbf{x}>\} \tag{41}$$

where the last line was written in view of Eq.(22). From here, in the limit : $\mathbf{f} = 0$ we reobtain $p(k,n)$ defined in Eq.(7). Thus, using classical partition function, Eq.(39), (discussed in the form of Exercises 2.27 and 2.28 in the book, Ref. [37], by Guillemin) and applying to it the Todd transform we recover the quantum mechanical partition function whose ground state provides us with the combinatorial factor $p(k,n)$.

5 From analysis to synthesis

5.1 The Poincare$'$ polynomial

The results discussed earlier are obtained for some fixed dilation factor k . In view of (6), they can be rewritten in the form valid for any dilation factor k. To this purpose it is convenient to rewrite (6) in the following equivalent form:

$$\frac{1}{\det(1-Mt)} = \frac{1}{(1-tz_0)\cdots(1-tz_k)} = (1+tz_0+(tz_0)^2+...)\cdots(1+tz_n+(tz_n)^2+...)$$
$$= \sum_{n=0}^\infty(\sum_{k_0+...+k_k=n} z_0^{k_0}\cdots z_k^{k_k})t^n \equiv \sum_{n=0}^\infty tr(M_n)t^n, \tag{42}$$

where the linear map from $k+1$ dimensional vector space V to V is given by matrix $M \in G \subset GL(V)$ whose eigenvalues are $z_0, ..., z_k$. Using this observation

[8]Mathematically inclined reader is encoraged to read paper by Brion and Vergne, Ref.[39], where all missing details are scrupulously presented.

several conclusions can be drawn. First, it should be clear that

$$\sum_{k_0+...+k_k=n} z_0^{k_0} \cdots z_k^{k_k} = \sum_{\mathbf{m} \in n\Delta \cap \mathbf{Z}^{k+1}} \mathbf{x^m} = tr(M_n). \tag{43}$$

Second, following Stanley [3,15] we would like to consider the algebra of invariants of G. To this purpose we introduce a basis $\mathbf{x} = \{ x_0, ..., x_k \}$ of V and the polynomial ring $R = \mathbf{C}[x_0, ..., x_k]$ so that if $f \in R$, then $Mf(\mathbf{x}) = f(M\mathbf{x})$. The algebra of invariant polynomials R^G can be defined now as

$$R^G = \{ f \in R : Mf(\mathbf{x}) = f(M\mathbf{x}) = f(\mathbf{x}) \ \ \forall M \in G \}.$$

These invariant polynomials can be explicitly constructed as averages over the group G according to prescription:

$$Av_G f = \frac{1}{|G|} \sum_{M \in G} Mf, \tag{44}$$

with $|G|$ being the cardinality of G. Suppose now that $f \in R^G$, then, evidently, $f \in R^G = Av_G f$ so that $Av_G^2 f = Av_G f = f$. Hence, the operator Av_G is indepotent. Because of this, its eigenvalues can be only 1 and 0. From here it follows that

$$\dim f_n^G = \frac{1}{|G|} \sum_{M \in G} tr(M_n). \tag{45}$$

Thus far our analysis was completely general. To obtain Eq.(7) we have to put $z_0 = ... = z_k = 1$ in (6). This time, however we can use the obtained results in order to write the following expansion for the Poincare$'$ polynomial [3,15,28] which for the appropriately chosen G is equivalent to (8):

$$P(R^G, t) = \sum_{n=0}^{\infty} \frac{1}{|G|} \sum_{M \in G} tr(M_n)t^n = \sum_{n=0}^{\infty} \dim f_n^G t^n. \tag{46}$$

Evidently, the Ehrhart polynomial $\mathfrak{P}(k, n) = \dim f_n^G$. To figure out the group G in the present case is easy since, actually, the group is trivial: $G = 1$.This is so because the eigenvalues $z_0, ..., z_k$ of the matrix M all are equal to 1. It should be clear, however, that for some appropriately chosen group G expansion (17) is also the Poincare polynomial (for the Cohen -Macaulay polynomial algebra [3,15,28]). This fact provides independent (of Refs. [9,10]) evidence that both the Veneziano and Veneziano-like amplitudes are of topological origin.

5.2 Connections with intersection theory

We would like to strengthen this observation now. To this purpose, in view of (23), and taking into account that for the symplectic 2-form $\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i$ the n-th power is given by $\Omega^n = \Omega \wedge \Omega \wedge \cdots \wedge \Omega = dx_1 \wedge dy_1 \wedge \cdots dx_n \wedge dy_n$, it is convenient to introduce the differential form

$$\exp \Omega = 1 + \Omega + \frac{1}{2!}\Omega \wedge \Omega + \frac{1}{3!}\Omega \wedge \Omega \wedge \Omega + \cdots \quad . \tag{47}$$

16

By design, the expansion in the r.h.s. will have only k terms. The form Ω is closed, i.e. $d\Omega = 0$ (the Liouvolle theorem), but not exact. In view of the expansion (47) the D-H integral (39) can be rewritten as

$$I(k, \mathbf{f}) = \int\limits_{k\Delta} \exp{(\tilde{\Omega})} \qquad (48)$$

where, following Atiyah and Bott [16] we have introduced the form $\tilde{\Omega} = \Omega - \mathbf{f} \cdot \mathbf{x}$. Doing so requires to replace the exterior derivative d acting on Ω by $\tilde{d} = d + i(\mathbf{x})$ (where the operator $i(\mathbf{x})$ reduces the degree of the form by one) with respect to which the form $\tilde{\Omega}$ is equivariantly closed, i.e. $\tilde{d}\tilde{\Omega} = 0$. More explicitly, we have $\tilde{d}\tilde{\Omega} = d\Omega + i(\mathbf{x})\Omega - \mathbf{f} \cdot \mathbf{dx} = 0$. Since $d\Omega = 0$, we obtain the equation for the moment map : $i(\mathbf{x})\Omega - \mathbf{f} \cdot \mathbf{dx} = 0$ [31,37]. If use of the operator d on differential forms leads to the notion of cohomology, use of the operator $\tilde{d}$ leads to the notion of equivariant cohomology. Although details can be found in the paper by Atiyah and Bott [16], more relaxed pedagogical exposition can be found in the monograph by Guillemin and Sternberg [40]. To make further progress, we would like to rewrite the two-form Ω in complex notations [31]. To this purpose, we introduce $z_j = p_j + iq_j$ and its complex conjugate. In terms of these variables Ω acquires the following form : $\Omega = \frac{i}{2} \sum_{i=1}^{k} dz_i \wedge d\bar{z}_i$. Next, recall [18] that for any Kähler manifold the fundamental 2-form Ω can be written as $\Omega = \frac{i}{2} \sum_{ij} h_{ij}(z)dz_i \wedge d\bar{z}_j$ provided that $h_{ij}(z) = \delta_{ij} + O(|z|^2)$. This means that in fact all Kähler manifolds are symplectic [14,34]. On such Kähler manifolds one can introduce the Chern cutrvature 2-form which (up to a constant) should look like Ω. It should belong to the first Chern class [19]. This means that, at least formally, consistency reqiures us to identify $x_i's$ entering the product $\mathbf{f} \cdot \mathbf{x}$ in the form $\tilde{\Omega}$ with the first Chern classes c_i, i.e. $\mathbf{f} \cdot \mathbf{x} \equiv \sum_{i=1}^{d} f_i c_i$. This fact was proven rigorously in the above mentioned paper by Atiyah and Bott [16]. Since in the Introduction we already mentioned that the Veneziano amplitudes can be formally associated with the period integrals for the Fermat (hyper)surfaces $\mathcal{F}$ and since such integrals can be interpreted as intersection numbers between the cycles on $\mathcal{F}$ [16,19] (see also Ref.[37], page72) one can formally rewrite the *precursor* to the Veneziano amplitude [10] as

$$I = \left(\frac{-\partial}{\partial f_0}\right)^{r_0} \cdots \left(\frac{-\partial}{\partial f_d}\right)^{r_d} \int\limits_{\Delta} \exp(\tilde{\Omega}) \mid_{f_i=0 \; \forall i} = \int\limits_{\Delta} d\mathbf{x}(c_0)^{r_0} \cdots (c_d)^{r_d} \qquad (49)$$

provided that $r_0 + \cdots + r_d = n$ in in view of Eq.(11). Analytical continuation of such an integral (as in the case of usual beta function) then will produce the Veneziano amplitudes. In such a language, calculation of the Veneziano amplitudes using generating function, Eq.(48), mathematically becomes almost equivalent to calculations of averages in the Witten-Kontsevich model [20-22]. In addition, as was also noticed by Atiyah and Bott [16], the replacement of the exterior derivative d by $\tilde{d} = d + i(\mathbf{x})$ was inspired by earlier work by Witten on supersymmetric formulation of quantum mechanics and Morse theory [17]. Such

an observation along with results of Ref.[40] allows us to develop calculations of the Veneziano amplitudes using supersymmetric formalism.

5.3 Supersymmetry and the Lefshetz isomomorphism

We begin with the following observations. Let X be the complex Hermitian manifold and let $\mathcal{E}^{p+q}(X)$ denote the complex -valued differential forms (sections) of type (p,q) , $p+q = r$, living on X. The Hodge decomposition insures that $\mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p+q}(X)$. The Dolbeault operators ∂ and $\bar{\partial}$ act on $\mathcal{E}^{p+q}(X)$ according to the rule $\partial : \mathcal{E}^{p+q}(X) \to \mathcal{E}^{p+1,q}(X)$ and $\bar{\partial} : \mathcal{E}^{p+q}(X) \to \mathcal{E}^{p,q+1}(X)$, so that the exterior derivative operator is defined as $d = \partial + \bar{\partial}$. Let now $\varphi_p, \psi_p \in \mathcal{E}^p$. By analogy with traditional quantum mechanics we define (using Dirac's notations) the inner product

$$< \varphi_p \mid \psi_p >= \int\limits_M \varphi_p \wedge *\bar{\psi}_p \tag{50}$$

where the bar means the complex conjugation and the star $*$ means the usual Hodge conjugation. Use of such a product is motivated by the fact that the period integrals, e.g. those for the Veneziano-like amplitudes, and, hence, those given by Eq.(49), are expressible through such inner products [19]. Fortunately, such a product possesses properties typical for the finite dimensional quantum mechanical Hilbert spaces. In particular,

$$< \varphi_p \mid \psi_q >= C\delta_{p,q} \text{ and } < \varphi_p \mid \varphi_p >> 0, \tag{51}$$

where C is some positive constant. With respect to such defined scalar product it is possible to define all conjugate operators, e.g. d^*, etc. and, most importantly, the Laplacians

$$\begin{aligned}
\Delta &= dd^* + d^*d, \\
\Box &= \partial\partial^* + \partial^*\partial, \\
\bar{\Box} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.
\end{aligned} \tag{52}$$

All this was known to mathematicians before Witten's work [17]. The unexpected twist occurred when Witten suggested to extend the notion of the exterior derivative d. Within the de Rham picture (valid for both real and complex manifolds) let M be a compact Riemannian manifold and K be the Killing vector field which is just one of the generators of isometry of M, then Witten suggested to replace the exterior derivative operator d by the extended operator

$$d_s = d + si(K) \tag{53}$$

briefly discussed earlier in the context of the equivariant cohomology. Here s is real nonzero parameter conveniently chosen. Witten argues that one can construct the Laplacian (the Hamiltonian in his formulation) Δ by replacing Δ by $\Delta_s = d_s d_s^* + d_s^* d_s$. This is possible if and only if $d_s^2 = d_s^{*2} = 0$ or, since

$d_s^2 = s\mathcal{L}(K)$, where $\mathcal{L}(K)$ is the Lie derivative along the field K, if the Lie derivative acting on the corresponding differential form vanishes. The details are beautifully explained in the much earlier paper by Frankel [41]. Atiyah and Bott observed that the auxiliary parameter **s** can be identified with earlier introduced **f**. This observation provides the link between the D-H formalism discussed earlier and Witten's supersymmetric quantum mechanics.

Looking at Eq.s (52) and following Ref.s[13,31,37] we consider the (Dirac) operator $\acute{\partial} = \bar{\partial} + \bar{\partial}^*$ and its adjoint with respect to scalar product, Eq.(50). Then use of above references suggests that the dimension Q of the quatum Hilbert space associated with the reduced phase space of the D-H integral considered earlier is given by

$$Q = \ker \acute{\partial} - co \ker \acute{\partial}^*. \tag{54}$$

Such a definition was also used by Vergne[13]. In view of the results of the previous section, and, in accord with Ref.[13], we make an identification: $Q = \mathfrak{P}(k, n)$.

We would like to arrive at this result using different set of arguments. To this purpose we notice first that according to Theorem 4.7. by Wells [18] we have $\Delta = 2\square = 2\bar{\square}$ with respect to the Kähler metric on X. Next, according to the Corollary 4.11. of the same reference Δ commutes with $d, d^*, \partial, \partial^*, \bar{\partial}$ and $\bar{\partial}^*$. From these facts it follows immediately that if we, in accord with Witten, choose Δ as our Hamiltonian, then the supercharges can be selected as $Q^+ = d + d^*$ and $Q^- = i(d - d^*)$. Evidently, this is not the only choice as Witten also indicates. If the Hamiltonian H is acting in *finite* dimensional Hilbert space one may require axiomatically that : a) there is a vacuum state (or states) $\mid \alpha >$ such that H$\mid \alpha >= 0$ (i.e. this state is the harmonic differential form) and $Q^+ \mid \alpha >= Q^- \mid \alpha >= 0$. This implies, of course, that $[$H$,Q^+] = [$H$,Q^-] = 0$. Finally, once again, following Witten, we may require that $(Q^+)^2 = (Q^-)^2 =$H. Then, the equivariant extension, Eq.(53), leads to $(Q_s^+)^2 =$ H$+2is\mathcal{L}(K)$. Fortunately, the above supersymmetry algebra can be extended. As it is mentioned in Ref.[19], there are operators acting on differential forms living on Kähler (or Hodge) manifolds whose commutators are isomorphic to $sl_2(\mathbf{C})$ Lie algebra. It is known [42] that *all* semisimple Lie algebras are made of copies of $sl_2(\mathbf{C})$. Now we can exploit these observations using the Lefschetz isomorphism theorem whose exact formulation is given as Theorem 3.12 in the book by Wells, Ref. [19]. We are only using some parts of this theorem in our work.

In particular, using notations of this reference we introduce the operator L commuting with Δ and its adjoint $L^* \equiv \Lambda$. It can be shown, Ref. [19], page 159, that $L^* = w * L*$ where, as before, $*$ denotes the Hodge star operator and the operator w can be formally defined through the relation $** = w$, Ref.[19] page 156. From these definitions it should be clear that L^* also commutes with Δ on the space of harmonic differential forms (in accord with page 195 of [19]). As part of the preparation for proving of the Lefschetz isomorphism theorem, it can be shown [19], that

$$[\Lambda, L] = B \text{ and } [B, \Lambda] = 2\Lambda, [B, L] = -2L. \tag{55}$$

At the same time, the Jacobson-Morozov theorem, Ref.[24], and results of Ref.[42], page 37, essentially guarantee that any $sl_2(\mathbf{C})$ Lie algebra can be brought into form

$$[h_\alpha, e_\alpha] = 2e_\alpha \ , \ [h_\alpha, f_\alpha] = -2f_\alpha \ , \ [e_\alpha, f_\alpha] = h_\alpha \tag{56}$$

upon appropriate rescaling. The index α counts number of $sl_2(\mathbf{C})$ algebras in a semisimple Lie algebra. Comparison between the above two expressions leads to the Lie algebra endomorphism, i.e. the operators h_α, f_α and e_α act on the vector space $\{v\}$ to be described below while the operators Λ, L and B obeying the same commutation relations act on the space of differential forms. It is possible to bring Eq.s(55) and (56) to even closer correspondence. To this purpose, following Dixmier [43], Ch-r 8, we introduce operators $h = \sum_\alpha a_\alpha h_\alpha$, $e = \sum_\alpha b_\alpha e_\alpha$, $f = \sum_\alpha c_\alpha f_\alpha$. Then, provided that the constants are subject to constraint: $b_\alpha c_\alpha = a_\alpha$, the commutation relations between the operators h, e and f are *exactly the same* as for B, Λ and L respectively. To avoid unnecessary complications, we choose $a_\alpha = b_\alpha = c_\alpha = 1$.

Next, following Serre, Ref. [23], Ch-r 4, we need to introduce the notion of the *primitive* vector (or element).This is the vector v such that $hv=\lambda v$ but $ev = 0$. The number λ is the weight of the module $V^\lambda = \{v \in V \mid hv=\lambda v\}$. If the vector space is *finite dimensional*, then $V = \sum_\lambda V^\lambda$. Moreover, only if V^λ is finite dimensional it is straightforward to prove that the primitive element does exist. The proof is based on the observation that if x is the eigenvector of h with weight λ, then ex is also the eigenvector of h with eigenvalue $\lambda - 2$, etc. Moreover, from the book by Kac [44], Chr.3, it follows that if λ is the weight of V, then $\lambda - < \lambda, \alpha_i^\vee > \alpha_i$ is also the weight with the same multiplicity, provided that $< \lambda, \alpha_i^\vee >\in \mathbf{Z}$. Kac therefore introduces another module: $U = \sum_{k\in\mathbf{Z}} V^{\lambda+k\alpha_i}$. Such a module is finite for finite Weyl-Coxeter reflection groups and is infinite for the affine reflection groups associated with the affine Kac-Moody Lie algebras.

We would like to argue that for our purposes it is sufficient to use only *finite* reflection (or pseudo-reflection) groups. It should be clear, however, from reading the book by Kac that the infinite dimensional version of the module U leads straightforwardly to all known string-theoretic results. In the case of CFT this is essential, but for calculation of the Veneziano-like amplitudes this is *not* essential as we are about to demonstrate.Indeed, by accepting the traditional option we loose at once our connections with the Lefschetz isomorphism theorem (relying heavily on the existence of primitive elements) and with the Hodge theory in its standard form on which our arguments are based. The infinite dimensional extensions of the Hodge-de Rham theory involving loop groups, etc. relevant for CFT can be found in Ref.[45]. Fortunately, they are not needed for our calculations. Hence, below we work only with the finite dimensional spaces.

In particular, let v be a primitive element of weight λ then, following Serre,

we let $v_n = \frac{1}{n!}e^n v$ for $n \geq 0$ and $v_{-1} = 0$, so that

$$hv_n = (\lambda - 2n)v_n \tag{57}$$
$$ev_n = (n+1)v_{n+1}$$
$$fv_n = (\lambda - n + 1)v_{n-1}.$$

Clearly, the operators e and f are the creation and the annihilation operators according to the existing in physics terminology while the vector v can be interpreted as the vacuum state vector. The question arises: how this vector is related to the earlier introduced vector $\mid \alpha >$? Before providing an answer to this question we need, following Serre, to settle the related issue. In particular, we can either: a) assume that for all $n \geq 0$ the first of Eq.s(57) has solutions and all vectors v, v_1, v_2 , ..., are linearly independent or b) beginning from some $m + 1 \geq 0$, all vectors v_n are zero, i.e. $v_m \neq 0$ but $v_{m+1} = 0$. The first option leads to the infinite dimensional representations associated with Kac-Moody affine algebras just mentioned. The second option leads to the finite dimensional representations and to the requirement $\lambda = m$ with m being an integer. Following Serre, this observation can be exploited further thus leading us to crucial physical identifications. Serre observes that with respect to $n = 0$ Eq.s(57) possess a ("super")symmetry. That is the linear mappings

$$e^m : V^m \to V^{-m} \text{ and } f^m : V^{-m} \to V^m \tag{58}$$

are isomorphisms and the dimensionality of V^m and V^{-m} are the same. Serre provides an operator (the analog of Witten's F operator) $\theta = \exp(f)\exp(e)\exp(-f)$ such that $\theta \cdot f = -e \cdot \theta$, $\theta \cdot e = -\theta \cdot f$ and $\theta \cdot h = -h \cdot \theta$. In view of such an operator, it is convenient to redefine h operator : $h \to \hat{h} = h - \lambda$. Then, for such redefined operator the vacuum state is just v. Since both L and $L^* = \Lambda$ commute with the supersymmetric Hamiltonian H and, because of the group endomorphism, we conclude that the vacuum state $\mid \alpha >$ for H corresponds to the primitive state vector v.

Now we are ready to apply yet another isomorphism following Ginzburg [24, Chap. 4, pages 205-206] [9]. To this purpose we make the following identification

$$e_i \to t_{i+1}\frac{\partial}{\partial t_i} \ , \ f_i \to t_i\frac{\partial}{\partial t_{i+1}} \ , \ h_i \to 2\left(t_{i+1}\frac{\partial}{\partial t_{i+1}} - t_i\frac{\partial}{\partial t_i}\right), . \tag{59}$$

$i = 0, ..., m$. Such operators are acting on the vector space made of monomials of the type

$$v_n \to \mathcal{F}_n = \frac{1}{n_0!n_1!\cdots n_k!}t_0^{n_0}\cdots t_k^{n_k} \tag{60}$$

where $n_0 + ... + n_k = n$. This result is useful to compare with Eq. (49).

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[9]Unfortunately, the original sourse contains minor mistakes. These are easily correctable. The corrected results are given in the text.

Eq.s (57) have now their analogs

$$h_i * \mathcal{F}_n(i) = 2(n_{i+1} - n_i)\mathcal{F}_n(i)$$
$$e_i * \mathcal{F}_n(i) = 2n_i\mathcal{F}_n(i+1) \tag{61}$$
$$f_i * \mathcal{F}_n(i) = 2n_{i+1}\mathcal{F}_n(i-1),$$

where, clearly, one should make the following consistent identifications: $m(i) - 2n(i) = 2(n_{i+1} - n_i)$, $2n_i = n(i) + 1$ and $m(i) - n(i) + 1 = 2n_{i+1}$. Next, we define the total Hamiltonian: $h = \sum_{i=0}^{k} h_i$ so that $\sum_{i=0}^{k} m(i) = n$, and then consider its action on one of the wave functions of the type given by Eq.(60).Since the operators defined by Eq.s (59) by design preserve the total degree of monomials of the type given by Eq.(60) (that is they preserve the Veneziano energy-momentum codition), we obtain the ground state degeneracy equal to $\mathfrak{P}(k,n)$ in agreement with Vergne, Ref. [13], where it was obtained using different methods. Clearly, the factor $\mathfrak{P}(k,n)$ is just the number of solutions in nonnegative integers to $n_0 + ... + n_k = n$, Ref.[33], page 252.

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